Matrices

In this chapter, the principals and algebra of matrices are presented in simple form together with several illustrative examples to clarify the presented concepts.

Basic Concepts

An ordered array A of $m \times n$ elements is called matrix and is written in the form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- The elements lying on a horizontal line form a row.
- The elements lying on a vertical line form a column.

Example 1

The matrix $A = \begin{bmatrix} 2 & 0 & 3 & -2 \\ 3 & 4 & 1 & -3 \end{bmatrix}$ of order 2x4

The matrix $B = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -3 & 4 \\ -1 & 12 & 9 \end{bmatrix}$ of order 3x3

The matrix $C=\begin{bmatrix} 2 & 0.2 \\ 1.3 & -1 \end{bmatrix}$ of order 2x2

The matrix $D=\begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 2 & 8 \end{bmatrix}$ of order 3x2

The form $\begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & \\ 2 & 0 & 5 \end{bmatrix}$ is not matrix

Remark 1

Let A and B be two matrices of the same order. Then A = B if all the corresponding elements are identical.

Example 2

The matrix A equals to the matrix B:

Where $A = \begin{bmatrix} 2 & 0 & 3 & -2 \\ 3 & 4 & 1 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 3 & -2 \\ 3 & 4 & 1 & -3 \end{bmatrix}$

Special Types of Matrices

- **Square Matrix**

Number of rows = Number of columns

Example 3

The matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ is square

Its diagonal is the numbers 1, 6

The matrix $B = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & 2 \\ 6 & 5 & 8 \end{bmatrix}$ is square

Its diagonal is the numbers 2, 3, 8

- **Row Matrix: It contains one row**

Example 4

$$A = \begin{bmatrix} 1 & 5 & -2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix}$$

- Column Matrix:

  It contains one column

Example

$$C = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

Null (Zero) Matrix

All elements are zeros

Example 5

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ is null matrix

$$O = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is null matrix

Upper Triangular Matrix

If all elements under the diagonal of a square matrix are zeros. Then it is called upper triangular matrix

Example 6

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Lower Triangular Matrix

If all elements above the diagonal of a square matrix are zeros. Then it is called lower triangular matrix

Example

$$A = \begin{bmatrix} 3 & 0 \\ 2 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 3 & 1 & 4 & 5 \end{bmatrix}$$

Identity (Unit) Matrix

If all elements of a square matrix are zeros and each element of diagonal is 1. Then, it is called identity matrix and denoted by I.

Example 7

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Algebra of Matrices

- **Transpose**

The transpose of a matrix A of order **mxn**, denoted by A\` or $A^t$ , is a matrix of order **nxm** where the row in A is the column in A\`.

Example 8

The transpose of $A = \begin{bmatrix} 2 & 5 & 1 \\ 3 & 1 & 8 \end{bmatrix}$

is the matrix $A` = \begin{bmatrix} 2 & 3 \\ 5 & 1 \\ 1 & 8 \end{bmatrix}$

The transpose of $B = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 4 \\ 3 & 4 & 1 \end{bmatrix}$

is the matrix $B` = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 4 \\ 3 & 4 & 1 \end{bmatrix}$

Multiplication by Scalar

If A is a matrix of order mxn and **k** is a scalar. Then kA is matrix of the same order where all elements of A are multiplied by k.

Example 10

If $A = \begin{bmatrix} 2 & 5 & 1 \\ 3 & 0 & 8 \end{bmatrix}$. Then $3A = \begin{bmatrix} 6 & 15 & 3 \\ 9 & 0 & 24 \end{bmatrix}$

Addition

Let A and B be two matrices of order mxn. Then A + B = C where each element of C is the sum of corresponding elements of A and B.

$$A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & . & . & . & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & . & . & . & a_{2n}+b_{2n} \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & . & . & . & a_{mn}+b_{mn} \end{bmatrix}$$

Example 11

If $A = \begin{bmatrix} 2 & 5 & 1 \\ 3 & 1 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & -2 & 0 \\ 1 & 3 & 8 \end{bmatrix}$

Then $A + B = \begin{bmatrix} 8 & 3 & 1 \\ 4 & 4 & 16 \end{bmatrix}$

$$A - B = \begin{bmatrix} -4 & 7 & 1 \\ 2 & -2 & 0 \end{bmatrix}$$

Example 12

If $A = \begin{bmatrix} 2 & 5 & 1 \\ 3 & 1 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 0 & 3 \\ 1 & 2 \end{bmatrix}$

Then $A + B$ does not exist.

$$A + B` = \begin{bmatrix} 2 & 5 & 1 \\ 3 & 1 & 8 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 1 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 2 \\ 4 & 4 & 10 \end{bmatrix}$$

Multiplication

Let A be a matrix of order **mxn** and B be a matrix of order **nxk**. Then, the multiplication of A and B, denoted by AB, is a matrix C of order mxk where the element in C is obtained by the sum of the product of the elements of the ith row of A and the elements of the jth column of B.

Row from A Column from B

$$\begin{bmatrix} a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{j1} \\ b_{j2} \\ \vdots \\ b_{jn} \end{bmatrix}$$

Then $c_{ij} = a_{i1}b_{j1} + a_{i2}b_{j2} + \dots + a_{in}b_{jn}$

= sum of product of elements of row from A with elements of column from B

Example 13

Find, if possible, AB, BA where

$$A = \begin{bmatrix} 2 & 5 & 1 \\ 3 & 1 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 3 & 1 \\ 2 & 0 & 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 5 & 1 \\ 3 & 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 2 & 3 & 1 \\ 2 & 0 & 4 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 2+10+2 & 4+15+0 & -4+5+4 \\ 3+2+16 & 6+3+0 & -6+1+32 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 19 & 5 \\ 21 & 9 & 27 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 3 & 1 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & 5 & 1 \\ 3 & 1 & 8 \end{bmatrix}$$

We see that does not exist

Example 15

Find AB, BA where

$$A = \begin{bmatrix} 2 & 5 & 1 \\ 3 & 1 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 1 & 4 \end{bmatrix}$$

Then $AB = \begin{bmatrix} 2 & 5 & 1 \\ 3 & 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 13 & 20 \\ 13 & 43 \end{bmatrix}$

Also

$$BA = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 5 & 1 \\ 3 & 1 & 8 \end{bmatrix} = \begin{bmatrix} 11 & 8 & 25 \\ 10 & 12 & 18 \\ 14 & 9 & 33 \end{bmatrix}$$

Determinants

The determinant of a square matrix A, denoted by $|A|$, is a scalar associated with the elements of the matrix A.

That is,

- The determinant of a matrix of one element is that element.
- The determinant of the null square matrix is zero.

$$\text{If} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \text{ Then } |A| = ad - bc.$$

$$\text{If} \quad B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}. \quad \text{Then } |B| = 12 - 2 = 10$$

$$\text{If} \quad C = \begin{bmatrix} 2 & -2 \\ 1 & -3 \end{bmatrix}. \text{ Then } |C| = -6 + 2 = -4$$

$$\text{If} \quad D = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1/2 \end{bmatrix}. \ |D| \text{ does not exist.}$$

Cofactors

If A is a square matrix of order n. The cofactor of an element a_{ij} which lies in the ith row and jth column is given by:

$$A_{ij} = (-1)^{i+j} M_{ij}$$

Where M_{ij} is the determinant obtained from the matrix A by eliminating the ith row and jth column

In general

If A is a square matrix of order n. Then

$|A|$ = sum of product of elements of any row (or any column) with its cofactors.

Example 18

$$\text{If}\quad A=\begin{bmatrix}1 & 2 & -1\\ 3 & 0 & 1\\ 2 & 3 & 4\end{bmatrix}$$

$$\text{Then}\quad |A| = 1\begin{vmatrix}0 & 1\\ 3 & 4\end{vmatrix} - 2\begin{vmatrix}3 & 1\\ 2 & 4\end{vmatrix} - 1\begin{vmatrix}3 & 0\\ 2 & 3\end{vmatrix}$$

$$= (0-3) - 2(12-2) - (9-0) = -32$$

Example

$$\text{If } A = \begin{bmatrix} -2 & -3 & 1 \\ 3 & 0 & 1 \\ 2 & -1 & 4 \end{bmatrix}$$

$$\text{Then } |A| = -2\begin{vmatrix} 0 & 1 \\ -1 & 4 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} + 1\begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix}$$

$$= -2(0 + 1) + 3(12 - 2) + (-3 - 0)$$

$$= 25$$

Properties

If A is a square matrix of order n. Then

- $|A`| = |A|$
- $|A| = 0$ if A has either zero row or zero column.
- $|A| = 0$ if A has two rows (columns) alike.
- $|B| = -|A|$ where B is a matrix obtained by interchanging two rows (columns) of A.
- The determinant of a triangular matrix

equals to the product of its diagonal elements.

Example 20

If $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 6 \end{bmatrix}$ which is triangular matrix

Then $|A| = 2.1.6 = 12$

If $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ which is triangular matrix

Then $|B| = 2.1.3 = 6$

$$\text{If} \quad C = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 3 \\ 1 & 8 & 6 \end{bmatrix} \quad \text{Then } |C| = 0$$

$$\text{If} \quad D = \begin{bmatrix} 5 & 1 & 3 \\ 0 & 1 & 3 \\ 1 & 8 & 8 \end{bmatrix} \quad \text{Then } |D| = 0$$

$$\text{If} \quad E = \begin{bmatrix} 2 & 1 & 5 & 2 \\ 0 & 0 & 0 & 0 \\ 3 & 1 & 4 & 2 \\ 1 & 3 & 5 & 0 \end{bmatrix} \quad \text{Then } |E| = 0$$

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$. Then $|A| = 8 - 6 = 2$

Also, $A` = \begin{bmatrix} 1 & 3 \\ 2 & 8 \end{bmatrix}$ and $|A`| = 8 - 6 = 2$

We see that $|A| = |A`|$

Inverse of Matrix

- A square matrix A is called singular if $|A| = 0$ and it is nonsingular matrix if $|A| \neq 0$.
- If A is nonsingular matrix, then it has inverse denoted by A^{-1} and is given by:

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A = \frac{1}{|A|} [\text{cofactors}]^{t}$$

Example 23

If $A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$. Then $|A| = 8 - 8 = 0$

It is singular matrix and has no inverse.

If $B = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 1 & 3 \end{bmatrix}$

Then $|B|$ does not exist and it has no inverse

Example 24

If $A = \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix}$. Then $|A| = 5 - 2 = 3$

It is nonsingular matrix and has inverse.

Then $A^{-1} = \frac{1}{3}\begin{bmatrix} 5 & -1 \\ -2 & 1 \end{bmatrix}^t = \frac{1}{3}\begin{bmatrix} 5 & -2 \\ -1 & 1 \end{bmatrix}$

If $B = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$.

Then $|B| = 2(4 - 9) + (2 - 3) + 0 = - 11$

It is nonsingular matrix and it has inverse.

Since $B = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$

Then

$$\operatorname{adj}B = \begin{bmatrix} \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \\ -\begin{vmatrix} -1 & 0 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} \\ \begin{vmatrix} -1 & 0 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} \end{bmatrix}^t$$

Then

$$\operatorname{adj}B = \begin{bmatrix} -5 & 1 & 1 \\ 2 & 4 & -7 \\ -3 & -6 & 5 \end{bmatrix}^t = \begin{bmatrix} -5 & 2 & -3 \\ 1 & 4 & -6 \\ 1 & -7 & 5 \end{bmatrix}$$

Then $B^{-1} = -\frac{1}{11}\begin{bmatrix} -5 & 2 & -3 \\ 1 & 4 & -6 \\ 1 & -7 & 5 \end{bmatrix}$

Eigenvalues

Many physical phenomena lead to linear algebraic equations:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = \lambda x_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = \lambda x_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = \lambda x_n$$

where λ is a parameter.

These equations can be written in matrix from:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Or $AX = \lambda X$

λ is called eigenvalue and X is called eigenvector

If A is a square matrix of order n, then its eigenvalues can be obtained by solving the equation: $|A - \lambda I| = 0$

It takes the form:

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0$$

This equation has n roots $\lambda_1, \lambda_2, \dots, \lambda_n$

For each eigenvalue λ we can obtain eigenvector X by solving the equation:

$$[A - \lambda I]X = 0$$

Example

Find the eigenvalues and eigenvectors of:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 2 = 0$$

Then $\lambda^2 - 3\lambda + 2 - 2 = 0$ Or $\lambda^2 - 3\lambda = 0$

The roots of this equation are 0, 3.

Then, the eigenvalues are 0, 3.

From the equation

$$\begin{bmatrix} 1-\lambda & 2 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

For $\lambda = 0$: $$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

Then $a + 2b = 0$ Or $a = -2b$

Putting $b = 1$, we get $a = -2$

Then, the eigenvector is: $X_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

From the equation

$$\begin{bmatrix} 1-\lambda & 2 \\ 1 & 2-\lambda \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0$$

For $\lambda = 3$: $$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0$$

Then $-2a + 2b = 0$ Or $a = b$

Putting $b = 1$, we get $a = 1$

Then, the eigenvector is: $$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example

Find the eigenvalues and eigenvectors of:

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 0 = 0$$

Then $(1-\lambda)(2-\lambda) = 0$

The roots of this equation are 1, 2.

Then, the eigenvalues are 1, 2.

From the equation

$$\begin{bmatrix} 1-\lambda & 3 \\ 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

For $\lambda = 1$: $$\begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

Then $0a + 3b = 0$.

Then $b = 0$, $a =$ any number

Putting $a = 1$, we get the vector is $X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

From the equation

$$\begin{bmatrix} 1-\lambda & 3 \\ 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

For $\lambda = 2$: $$\begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

Then $-a + 3b = 0$ Or $a = 3b$

Putting $b = 1$, we get $a = 3$

Then, the eigenvector is: $X_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Example 30

Find the eigenvalues and eigenvectors of:

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 3 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - 12 = 0$$

Then $\lambda^2 - 5\lambda - 6 = 0$

The roots of this equation are -1, 6.

Then, the eigenvalues are -1, 6.

From the equation

$$\begin{bmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

For $\lambda = -1$: $\begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$

Then $3a + 4b = 0$ Or $a = -4b/3$

Putting $b = -3$, we get $a = 4$

Then, the eigenvector is $X_1 = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

From the equation

$$\begin{bmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0$$

For $\lambda = 6$: $\begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0$

Then $-4a + 4b = 0$ Or $a = b$

Putting $b = 1$, we get $a = 1$

Then, the eigenvector is: $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thank You